Radioactive Decay: Concepts And Mathematics

RADIOACTIVE DECAY: CONCEPTS AND MATHEMATICAL APPLICATIONS

Radioactive Decay

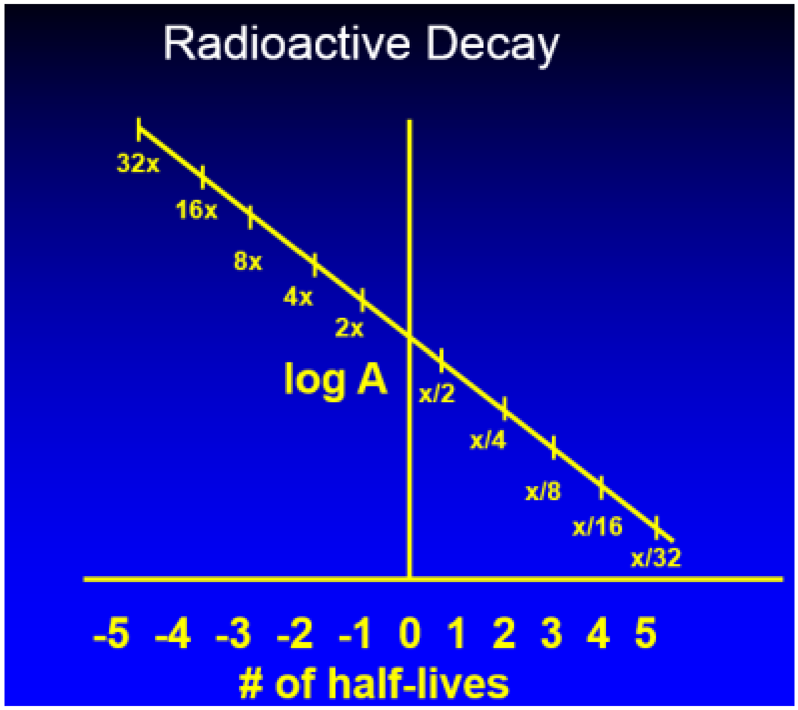
1. *Decay* follows an exponential law and is described in terms of *half-life*, the time required for one half of any starting amount of an unstable radionuclide to undergo nuclear rearrangement and to produce the daughter radionuclide.

2. The decay of a particular radionuclide is therefore characterized by the type of emission, the energy of the emissions, and by the physical half-life.

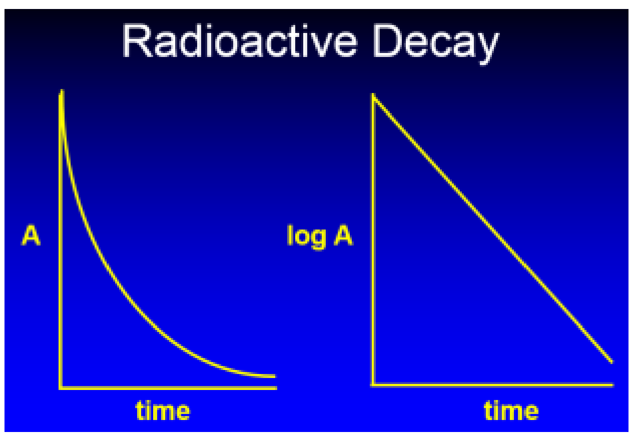
3. After one half-life, 1/2 of the starting material will be left; after two half-lives, 1/4 of the starting material will be left; after three half-lives, 1/8 of the starting material will be left; refer to graph below

4. One half-life ago, 2x as much of the starting material was present; two half-lives ago, 4x as much was present; three half-lives ago, 8x as much was present; refer to graph below

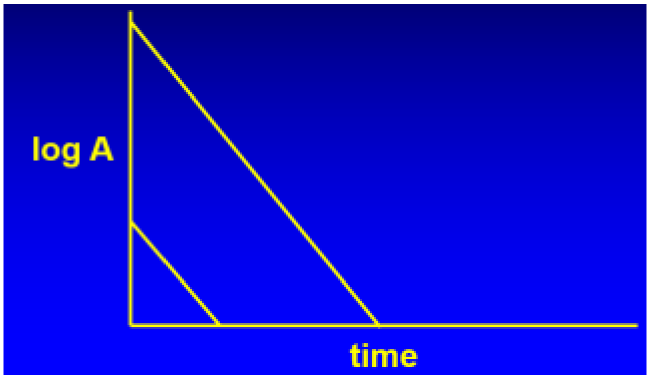
5. Graph of radioactivity as a function of elapsed time



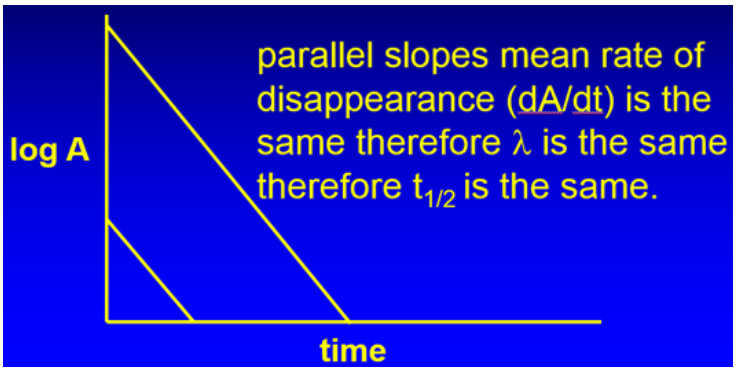
6. Decay plotted on both linear paper (left curve) and semilog paper (right curve). Reading values off either curve will yield same values, but decay is logarithmic and therefore it is preferable to plot time/activity curves on semilog paper.



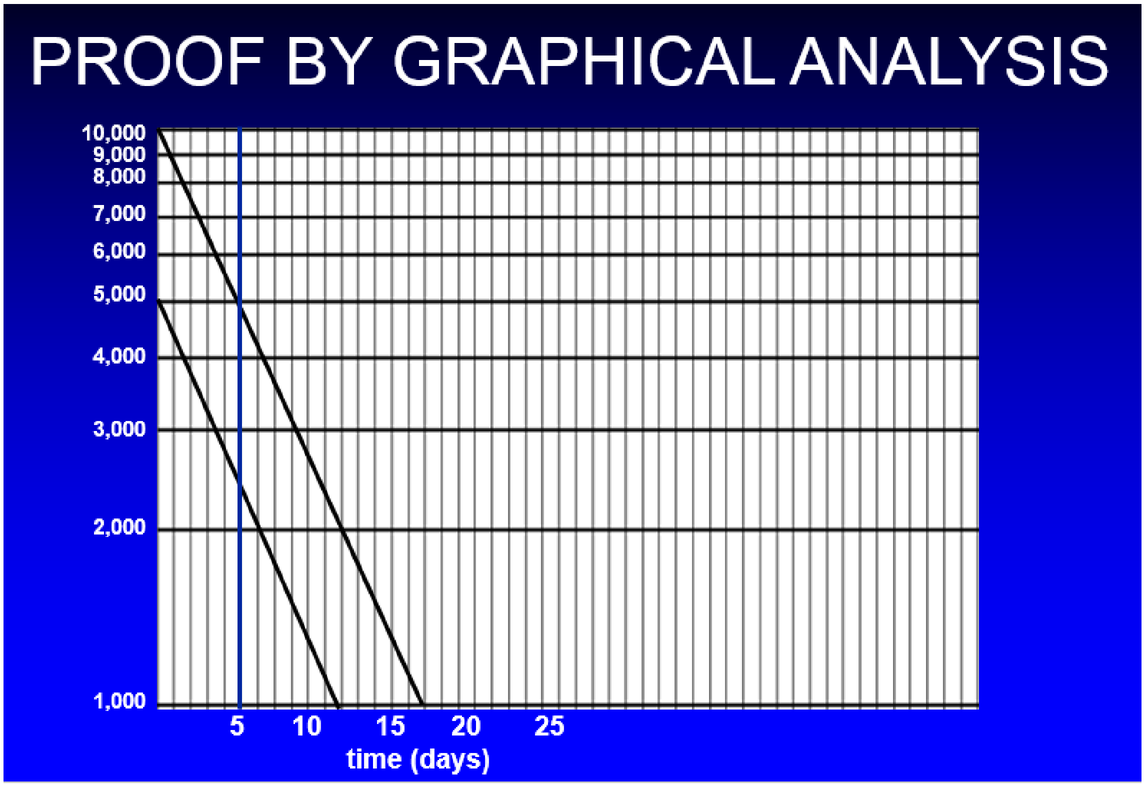
7. QUIZ: All parallel lines on a decay curve have the same $t_{1/2}$. True/False?



ANSWER: TRUE. All parallel lines on a decay curve DO have the same $t_{1/2}$.



8. Proof by graphical analysis:



For either line, the $t_{1/2}$ = 5 days.

For the curve at right, at time 0, the activity is 10,000 cpm. At the half-life, activity is 5,000 cpm which intersects the curve at 5 days.

For the curve at left, at time 0, the activity is 5,000 cpm. At the half-life, activity is 2,500 cpm which intersects the curve at 5 days.

Basically, one sample is “hotter” than the other, but the fraction decaying per unit of time is identical for both and therefore the half-lives must be equal.

Radioactive Decay: The Classic Mathematical Formula is

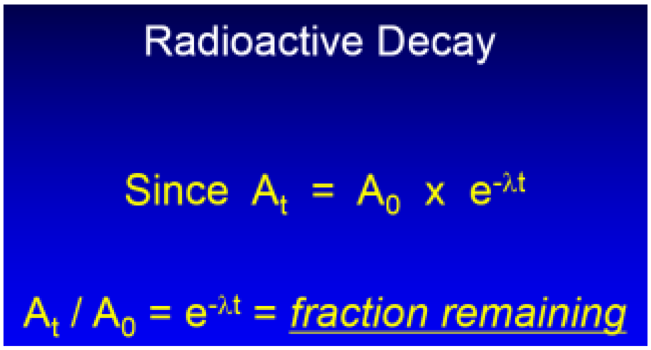
$A_t = A_0 \times e^{-\lambda t}$ where

A_t = activity at any point in time

A_0 = activity at time 0

λ = decay constant

t = elapsed time



What's important to remember is: $e^{-\lambda t}$ = ***fraction remaining***

Radioactive Decay: Simple, Logical Formula

For Prospective Radioactive Decay (going forward in time)

After 0 half-lives, $1/2^{(0)}$ remains

After 1 half-lives, $1/2^{(1)}$ remains

After 2 half-lives, $1/2^{(2)}$ remains

After 3 half-lives, $1/2^{(3)}$ remains

After 5 half-lives, $1/2^{(5)}$ remains

After 8 half-lives, $1/2^{(8)}$ remains

By extrapolation,

After n half-lives, $1/2^{(n)}$ remains

What's important to remember is: 0.5^n = ***fraction remaining***

Since

0.5^{n} = *fraction remaining* and $e^{-\lambda t}$ = *fraction remaining*

Therefore $0.5^{n} = e^{-\lambda t}$

By substitution, $A_t = A_0 \times 0.5^n$ where n = # of half-lives = $t_{elapsed} / t_{1/2}$

Therefore,

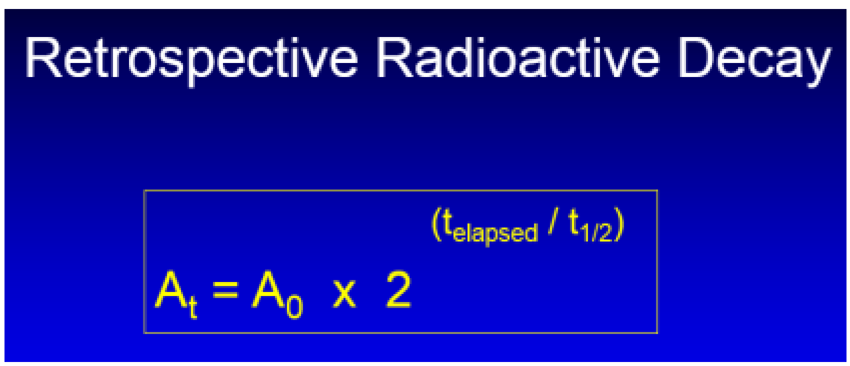
$$\boxed{A_t = A_0 \times 0.5^{(t_{elapsed} / t_{1/2})}}$$

There is no half-life problem that cannot be solved with this simple, logical equation. Answer will be identical to that obtained using the equation $A_t = A_0 \times e^{-\lambda t}$

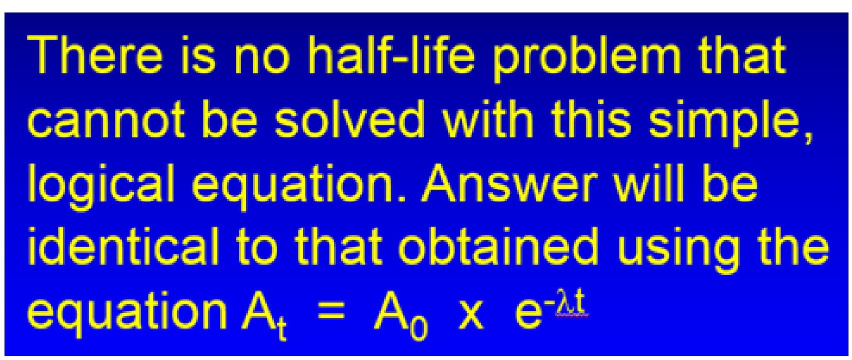
One of the distinct advantages of the equation $A_t = A_0 \times 0.5^{n}$ is that we humans are linear thinkers and therefore multiplying 0.5 by itself n times is logical and easy to understand. It is not at all intuitive that $e^{-\lambda t}$ is fraction remaining and not easy to perform the calculations, even with a scientific calculator.

Using a simple, logical argument for Retrospective Decay:

Therefore,



Again,



Sample problems

Problem 1. For Tc-99m, assume the current activity is 100 mCi. What will be the activity in 3 hr?

Solution:

$$A_t = A_0 \times 0.5^{(t_{elapsed}/t_{1/2})}$$

$$= 100 \text{ mCi} \times 0.5^{(3/6)}$$

Keystrokes: 100 x 0.5 $\boxed{Y^x}$ (3 ÷ 6) =

A_t = 70.71 mCi

Notes:

1. On some calculators, the Y^x key is represented by X^y or the symbol ^
2. You must always press "=" at end of calculation or incorrect answer will be obtained.

Problem 2. Tc-MDP was made at 7 A.M. and at that time, the radioconcentration was 75 mCi/ml. What volume must be drawn at 9:45 A.M. to yield a 25 mCi dose?

Solution

$$A_t = A_0 \times 0.5^{(t_{elapsed}/t_{1/2})}$$

A_t = 75 mCi/ml x $0.5^{(2.75/6)}$ = 54.59 mCi/ml at 9:45 AM

Keystrokes: 75 x 0.5 Y^x (2.75 ÷ 6) = 54.59 mCi/ml

Volume = Dose/Concentration = 25 mCi/ (54.59 mCi/ml) = 0.46 ml

Problem 3. A radioisotope was counted at two points in time. The initial reading was 20,000 cpm. Fourteen hr later, the count rate had dropped to 14,500. What is the half-life of this isotope?

Solution:

$$A_t = A_0 \times 0.5^{(t_{elapsed}/t_{1/2})}$$

$14{,}500 = 20{,}000 \times 0.5^{(14/HL)}$

$0.725 = 0.5^{(14/HL)}$ taking the logarithm of both sides,

$\log 0.725 = 14/HL \times \log 0.5$

$HL = 14 \times \log 0.5/\log 0.725 =$ 30.18 hr

Keystrokes: 14 x 0.5 log ÷ 0.725 log =

Problem 4. True or False: After half of a half-life, 75% of the initial activity remains. Why?

Answer:

False. The relationship is exponential. Linear extrapolation is not valid. In fact, the fraction remaining after half of a half-life is

$$\text{Fr Rem} = 0.5^n = 0.5^{0.5}$$

$$\text{Fr Rem} = 0.7071$$

Problem 5. Assume that the t_{biol} of Xe-gas in the lungs is 15 sec. What would be the residual activity in the lungs 1 minute after inhalation of 15 mCi of Xe-133 gas by a patient?

Answer: assume t_{biol} =15 sec; ∴ t_{eff} = 15 sec since t_{phys} (5.3 d) is very long compared to duration of study.

since 1 min = 4 t_{eff}, then for 4 t_{eff},

Fr Rem = $0.5^n = 0.5^4 = 0.0625$

$A_t = A_0 \times 0.0625$ = 15 mCi x 0.0625 = **0.94 mCi**

Problem 5a

if t_{eff} = 15 sec, then A_t = 0.94 mCi 1 minute after inhalation of 15 mCi of Xe-133 gas

if t_{eff} = 20 sec, then A_t = _______ 1 minute after inhalation of 15 mCi of Xe-133 gas

if t_{eff} = 30 sec, then A_t = _______1 minute after inhalation of 15 mCi of Xe-133 gas

Solution:

if t_{eff} = 15 sec, then A_t = 0.94 mCi since there are 4 t_{eff} in 1 minute

if t_{eff} = 20 sec, then A_t = 1.88 mCi since there are 3 t_{eff} in 1 minute

if t_{eff} = 30 sec, then A_t = 3.76 mCi since there are 2 t_{eff} in 1 minute

Problem 6. We hold radioactive waste for 10 half-lives before discarding it. What fraction of the original activity remains?

Solution:

Fr Rem = $0.5^n = 0.5^{10}$ so

Fr Rem = 0.000976

OR

Fr Rem = $(1/2)^{10}$ so

Fr Rem = 1/1024

Problem 6a (assume that after 10 half-lives, 1/1,000 remains, not 1/1,024, the correct value)

if 10 t_{phys} have elapsed, fr rem =1/1,000

if 20 t_{phys} have elapsed, fr rem = _____

if 30 t_{phys} have elapsed, fr rem = _____

if 40 t_{phys} have elapsed, fr rem = _____

ANSWER

if 10 t_{phys} have elapsed, fr rem = 0.5^{10} = $\sim 1 \times 10^{-3}$

if 20 t_{phys} have elapsed, fr rem = 0.5^{20} = $\sim 1 \times 10^{-6}$

if 30 t_{phys} have elapsed, fr rem = 0.5^{30} = $\sim 1 \times 10^{-9}$

if 40 t_{phys} have elapsed, fr rem = 0.5^{40} = $\sim 1 \times 10^{-12}$

Implication: if we place 1 mCi in the waste bin, after 10, 20, 30, and 40 half-lives, respectively, activity will decrease from a mCi to a µCi to a nCi to a pCi.

Problem 7. A sample of I-131 has an activity of 10 mCi. What was the activity 40 days ago?

Answer: (this problem involves retrospective decay)

40 days represents 5 half-lives of I-131. For retrospective decay,

$$\text{Fr Rem} = 2^n = 2^5 \qquad \text{so}$$

$A_t = 10 \text{ mCi} \times 2^5 = 10 \text{ mCi} \times 32$ and

$$A_t = 320 \text{ mCi}$$

Problem 8. The decay constant of an isotope is 0.0693 hr^{-1}. The original activity was 100 mCi. What is the activity after 10 hours of decay?

Answer:

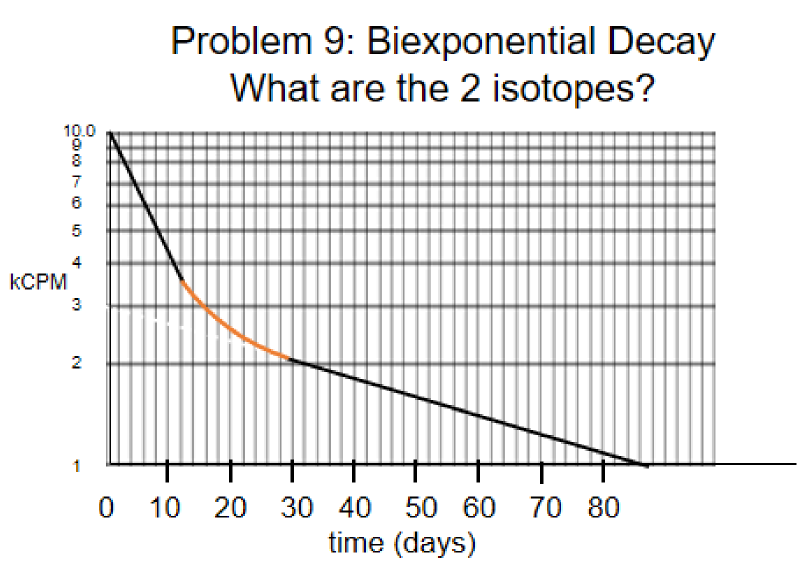
Since $\lambda = 0.693/t_{1/2}$

Then $t_{1/2} = 0.693/\lambda = 0.693/0.0693 \text{ hr}^{-1} = 10 \text{ hr}$

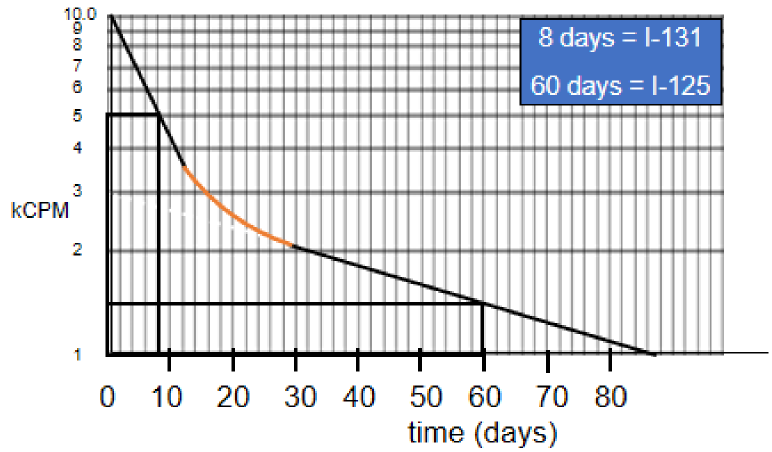
Therefore, 1 $t_{1/2}$ has elapsed and

$$A = 100 \text{ mCi} \times 0.5 = 50 \text{ mCi}$$

Problem 9.



Answer



Problem 10. A vial contains a mixture of 20 mCi of I^{124} (half life 4 days) and 6 mCi of I^{131} (half life 8 days). What will be the activity in the vial 8 days from now?

a) 5 mCi
b) 3 mCi
c) 8 mCi
d) 13 mCi

Answer: Correct answer is "c".

Activity = $(20 \times 0.5^{8/4}) + (6 \times 0.5^{8/8})$
= 5 + 3 = 8 mCi

so total activity = 8 mCi

Problem 11. At some point in time a source has an activity of 1,000 mCi. At a later point in time the activity is 62.5 mCi. The half-life is unknown. How many half-lives have elapsed?

a) three
b) four
c) five
d) can't be determined from data provided

Answer: Correct answer is “b”.

This is a pattern recognition problem. Logically, 1000>500>250>125>62.5 so 4 Half-lives have elapsed.

Mathematically,

Fraction remaining = 62.5/1000 = 0.0625 = 0.5^n
where n = # of half-lives.
Taking the log of both sides,
log 0.0625 = n x log 0.5 and
n = log 0.0625/log 0.5
therefore n = 4

Problem 12. For Tc-99m, what fraction remains after 19 hours?

a) 6/19
b) $(6/19)^2$
c) $0.5^{(6/19)}$
d) $0.5^{(19/6)}$

Problem 12. Correct answer is “d”.

Fraction remaining = $0.5^{\text{\# of half-lives}} = 0.5^{(19/6)}$